Polynomial Division

Example

Factor $x^2 - 5x + 6$

$$\begin{aligned} x^2 - 5x + 6 &= (x - \triangle)(x - \square) \\ &= (x-2)(x-3) \end{aligned}$$

Example

Factor $x^3 - 3x + 2$

$$\begin{aligned} x^3 - 3x + 2 &= (x - \triangle)(x^2 + ?x - \square) \\ &= (x-1)(x^2 + x - 2) \\ &= (x-1)(x-1)(x+2) \end{aligned}$$

Example

If $4x^3 + 2x^2 + 3 \equiv (x-2)(Ax^2 + Bx + C) + R$, find A, B, C and R

Method 1:

$$4x^3 + 2x^2 + 3 \equiv Ax^3 + (B - 2A)x^2 + (C - 2B)x - 2C + R$$

$$\Rightarrow \quad \begin{cases} A &= 4 \\ B - 2A &= 2 \\ C - 2B &= 0 \\ R - 2C &= 3 \end{cases}$$

$$\Rightarrow \quad A = 4$$

$$\Rightarrow \quad B = 2 + 8 = 10$$

$$\Rightarrow \quad C = 20$$

$$\Rightarrow \quad R = 2 \cdot 20 + 3 = 43$$

Method 2:

$$
\begin{aligned}
4x^3 + 2x^2 + 3 &= \underbrace{4(x-2)x^2 + 2 \cdot 4x^2}_{\text{add back the leftover}} + 2x^2 + 3 \\
&= 4x^2(x-2) + 10x^2 + 3 \\
&= 4x^2(x-2) + \underbrace{10x(x-2) + 20x}_{\text{add back extra}} + 3 \\
&= (4x^2 + 10x)(x-2) + \underbrace{20(x-2) + 40}_{\text{add back}} + 3 \\
&= (4x^2 + 10x + 20)(x-2) + 43
\end{aligned}
$$

Theorem (Polynomial Division)

If we have polynomials $f(x)$ and $g(x)$ of **degree** n and m, then we can write $f(x) = g(x)q(x) + r(x)$ where $q(x)$ has degree $n-m$ and $r(x)$ has degree less than m.

*What does the **degree** of a polynomial mean?*

Proof:

Apply the division method we've just used. If you want a challenge, try to prove this rigorously once you've studied induction in Pure.

Factor Theorem

Theorem (Factor Theorem)

For a polynomial $f(x)$, $f(a) = 0 \iff (x-a)$ is a factor of $f(x)$

Proof:

$(\Leftarrow)$: [always start with the easy direction]

If $(x-a)$ is a factor of $f(x)$ then $f(x) = (x-a)q(x)$, and so $f(a) = (a-a)q(a) = 0 \cdot q(a) = 0$, therefore $f(a) = 0$

$(\Rightarrow)$: Using polynomial division, write $f(x) = (x-a)q(x) + r(x)$, where $r(x)$ is a degree 0 (or lower) polynomial. (ie it is a constant). Then

$$f(a) = (a-a)q(a) + r(a) = 0 \cdot q(a) + r(a) = r(a)$$

But remember that $f(a) = 0 \Rightarrow r(a) = 0$, in particular $r = 0$, so $f(x) = (x-a)q(x)$ so $(x-a)$ is a factor!

Tip

Make sure you're using the correct direction!

Theorem (Remainder Theorem)

For a polynomial $f(x), f(a) = r \iff f(x) = (x-a)q(x) + r$ for some polynomial $q(x)$. Said another way, when $f(x)$ is divided by $x-a$ the remainder is $f(a)$.

By the division algorithm, we can write $f(x) = (x-a)q(x) + r$ for some polynomial $q(x)$ and constant r (since dividing by a linear polynomial leaves a constant remainder).

Substituting $x = a$: $f(a) = (a-a)q(a) + r = 0 + r = r$.

Therefore the remainder when $f(x)$ is divided by $(x-a)$ is $f(a)$.

Note: The Factor Theorem is the special case where $r = 0$, i.e., $(x-a)$ is a factor iff $f(a) = 0$.

Tip

We've not said whether or not the roots are integers, rationals or reals. One useful form is sometimes, if $f\left(\frac{p}{q}\right) = 0 \Leftrightarrow f(x) = (qx-p)g(x)$

Example

Factorise *completely* $x^4 - 3x^3 + 4x^2 - 8$

Let $f(x) = x^4 - 3x^3 + 4x^2 - 8$,
then $f(1) = 1 - 3 + 4 - 8 \neq 0$
but $f(-1) = 1 + 3 + 4 - 8 = 0$, *so* $x + 1$ *is a factor.*
$f(x) = (x+1)(x^3 - 4x^2 + 8x - 8)$. *So we can repeat the process with* $g(x) = x^3 - 4x^2 + 8x - 8$. *We don't need to check* $g(1)$, *but we also find* $g(2) = 0$. *(What should we be checking? Recall question on the quadratics homework sheet).*
Therefore $f(x) = (x+1)(x-2)(x^2 - 2x + 4)$.
Over $\mathbb{C}$, *we can factor further:* $x^2 - 2x + 4 = 0$ *gives* $x = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$.
So $f(x) = (x+1)(x-2)(x - 1 - i\sqrt{3})(x - 1 + i\sqrt{3})$.

Example

Solve the equation

$$(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$$

Notice that if we say $f(x) = x^2 - 3x + 3$, *we are looking at* $f(f(x)) = x$.
Clearly some solutions of this will come from $f(x) = x$ *(since* $f(x) = x \Rightarrow f(f(x)) = f(x) = x$*).*
Therefore we immediately see $x^2 - 3x + 3 = x \Rightarrow x^2 - 4x + 3 = (x-3)(x-1) \Rightarrow x = 1, 3$.
This means we must have a factor of $(x-1)(x-3) = x^2 - 4x + 3$ *to our polynomial:*

$$\begin{aligned}
0 &= (x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 - x \\
&= (\boxed{x^2 - 4x + 3} + x)^2 - 3(\boxed{x^2 - 4x + 3} + x) + 3 - x \\
&= (\boxed{x^2 - 4x + 3})^2 + 2x(\boxed{x^2 - 4x + 3}) + x^2 - 3(\boxed{x^2 - 4x + 3}) - 3x + 3 - x \\
&= (\boxed{x^2 - 4x + 3})^2 + (\boxed{x^2 - 4x + 3})(2x - 3) + (\boxed{x^2 - 4x + 3}) \\
&= (\boxed{x^2 - 4x + 3})(\boxed{x^2 - 4x + 3} + 2x - 3 + 1) \\
&= (\boxed{x^2 - 4x + 3})(x^2 - 2x + 1) \\
&= (x^2 - 4x + 3)(x - 1)^2
\end{aligned}$$

So there are no more solutions.
Think about other ways we could have done this polynomial division